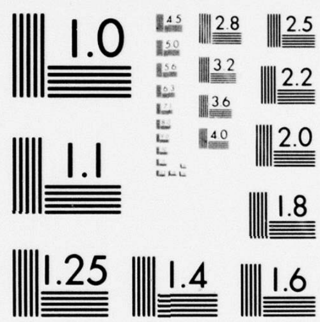
AD-A060 453 LEHIGH UNIV BETHLEHEM PA CENTER FOR THE APPLICATION--ETC F/G 20/11
THE INCREMENTAL SHEAR MODULUS IN AN INCOMPRESSIBLE ELASTIC MATE--ETC(U)
SEP 78 K N SAWYERS, R S RIVLIN N00014-76-C-0235
UNCLASSIFIED CAM-100-32 NL

| OF |
AD
A060 453

END
DATE
FILMED
1-79
DDC



MICROCOPY RESOLUTION TEST CHART

Lehigh University

LEVEL II

THE INCREMENTAL SHEAR MODULUS IN AN INCOMPRESSIBLE ELASTIC MATERIAL

by

K.N. SAWYERS [redacted] R.S. RIVLIN

20p.

TECHNICAL REP[redacted]T NO. CAM-100-32 Sep[redacted]1978

OFFICE OF NAVAL RESEARCH CONTRACT NO. N00014-76-C-0235

Center for the Application of Mathematics (CAM)

403 812 mt

The Incremental Shear Modulus in an

Incompressible Elastic Material

by

K.N. Sawyers and R.S. Rivlin

Center for the Application of Mathematics

Lehigh University, Bethlehem, Pa. 18015

Abstract

The incremental shear modulus is calculated for infinitesimal shear of an incompressible isotropic elastic material in a state of pure homogeneous deformation. Necessary and sufficient conditions that it be positive are obtained. These conditions are related to the Hadamard conditions for propagation of a wave of infinitesimal amplitude in the deformed material.

1. Introduction

In the present paper we calculate the incremental shear modulus for infinitesimal simple shear of an incompressible isotropic elastic material which is initially subjected to a pure homogeneous deformation, the direction of shear and plane of shear being arbitrary. (An analogous calculation has been carried out previously [1] for the case when the plane of shear is a principal plane for the pure homogeneous deformation.)

Necessary and sufficient conditions are obtained for the incremental shear modulus to be positive for all directions of shear perpendicular to a fixed direction, which lies in the plane of shear and is defined by a unit vector $\underset{\sim}{n}$ say. These are the inequalities (4.8) of the present paper. It is shown in §5 that they are equivalent to the necessary and sufficient conditions, previously obtained in [2] and given in (6.5) of the present paper, that the two sinusoidal waves of infinitesimal amplitude which can propagate in the deformed elastic material, in the direction $\underset{\sim}{n}$, have real velocities. The velocities of propagation of these waves are given (see equation (5.6) below) by $\sqrt{\mu/\rho}$, where ρ is the material density and μ is one or other of the stationary values of the incremental shear modulus for shearing in a direction perpendicular to $\underset{\sim}{n}$. It is shown in §7 that the directions of polarization for the waves are the directions for which the incremental shear modulus takes its stationary values.

The necessary and sufficient conditions (6.5) for the wave velocities to be real, and the equivalent conditions (4.8) for the incremental shear moduli to be positive, involve the first and second derivatives of the strain-energy function for the material with respect to the strain invariants, as well as the principal extension ratios for the pure homogeneous deformation and the unit vector $\underset{\sim}{n}$. It is desirable to obtain these conditions in a form which is independent of $\underset{\sim}{n}$. This has been achieved in [1,3] in the case when $\underset{\sim}{n}$ is restricted to lie in a principal plane of the pure homogeneous deformation. It has already been shown that the conditions so obtained (the inequalities (6.10) of the present paper) imply the condition $(6.5)_2$. In the present paper it is shown that a further one of the conditions (condition $(6.5)_3$) is automatically satisfied. It was shown in a previous paper [4], by numerical example, that the first of the conditions (6.5) is not implied by the conditions (6.10).

2. Incremental constitutive equation

We consider a deformation of an incompressible isotropic elastic material in which a particle which is initially at vector position $\underset{\sim}{\xi}$ with respect to a fixed origin 0 , say, moves to vector position $\underset{\sim}{x}$. The components of $\underset{\sim}{\xi}$ and $\underset{\sim}{x}$ in a fixed rectangular cartesian system x , with origin at 0 , are ξ_α and x_i respectively. The deformation gradient matrix $\underset{\sim}{\bar{g}}$ is defined by*

$$\underset{\sim}{\bar{g}} = \|\bar{g}_{i\alpha}\| = \|x_{i,\alpha}\| . \tag{2.1}$$

The Finger strain $\underset{\sim}{\bar{C}}$ is defined by

$$\underset{\sim}{\bar{C}} = \bar{g}\bar{g}^{\dagger} , \tag{2.2}$$

where the dagger denotes the transpose. Let W be the strain-energy per unit volume. Then, since the material is isotropic and incompressible, W must be expressible as a function of the invariants $\bar{I}_1$ and $\bar{I}_2$ of $\underset{\sim}{\bar{C}}$, defined by

$$\bar{I}_1 = \mathrm{tr}\,\underset{\sim}{\bar{C}} , \quad \bar{I}_2 = \tfrac{1}{2}\{(\mathrm{tr}\,\underset{\sim}{\bar{C}})^2 - \mathrm{tr}\,\underset{\sim}{\bar{C}}^2\} . \tag{2.3}$$

Thus,

$$W = W(\bar{I}_1, \bar{I}_2) . \tag{2.4}$$

The Cauchy stress matrix $\underset{\sim}{\bar{\sigma}} = \|\bar{\sigma}_{ij}\|$ is then given by

$$\underset{\sim}{\bar{\sigma}} = 2\left[\left(\frac{\partial W}{\partial \bar{I}_1} + \bar{I}_1 \frac{\partial W}{\partial \bar{I}_2}\right)\underset{\sim}{\bar{C}} - \frac{\partial W}{\partial \bar{I}_2}\,\underset{\sim}{\bar{C}}^2\right] - \bar{P}\underset{\sim}{\delta} , \tag{2.5}$$

where $\underset{\sim}{\delta}$ is the unit matrix and $\bar{P}$ is an arbitrary hydrostatic pressure.

---

*We use the notation $_{,\alpha}$ to denote differentiation with respect to ξ_α and shall assume that the Einstein summation convention applies to lower case Greek and Latin subscripts.

We now consider that the deformation $\underset{\sim}{\xi} \to \underset{\sim}{x}$ is the resultant of a finite deformation, in which the particle initially at vector position $\underset{\sim}{\xi}$ moves to vector position $\underset{\sim}{X}$, followed by an infinitesimal deformation in which it moves to vector position $\underset{\sim}{x}$. We shall accordingly write

$$\underset{\sim}{x} = \underset{\sim}{X} + \underset{\sim}{u} \tag{2.6}$$

and correspondingly

$$\begin{aligned} \bar{\underset{\sim}{\sigma}} = \underset{\sim}{\sigma} + \underset{\sim}{s}\ , \quad \bar{I}_1 = I_1 + i_1\ , \quad \bar{I}_2 = I_2 + i_2\ , \\ \bar{\underset{\sim}{C}} = \underset{\sim}{C} + \underset{\sim}{c}\ , \quad \bar{P} = P + p\ , \end{aligned} \tag{2.7}$$

where $\underset{\sim}{\sigma}$, I_1, I_2, $\underset{\sim}{C}$ and P are the values of $\bar{\underset{\sim}{\sigma}}$, $\bar{I}_1$, $\bar{I}_2$, $\bar{\underset{\sim}{C}}$ and $\bar{P}$ associated with the deformation $\underset{\sim}{\xi} \to \underset{\sim}{X}$.

With this notation, it follows from (2.5) that

$$\begin{aligned} \underset{\sim}{s} = 2\Big[&(W_1 + I_1 W_2)\underset{\sim}{c} - W_2(\underset{\sim}{C}\underset{\sim}{c} + \underset{\sim}{c}\underset{\sim}{C}) + i_1 W_2 \underset{\sim}{C} \\ &+ \{(W_{11} + I_1 W_{12})i_1 + (W_{12} + I_1 W_{22})i_2\}\underset{\sim}{C} - (W_{12}i_1 + W_{22}i_2)\underset{\sim}{C}^2\Big] - p\underset{\sim}{\delta} \end{aligned} \tag{2.8}$$

where

$$W_1 = \left.\frac{\partial W}{\partial \bar{I}_1}\right|_{\underset{\sim}{u}=0}\ , \quad W_2 = \left.\frac{\partial W}{\partial \bar{I}_2}\right|_{\underset{\sim}{u}=0}\ , \quad W_{11} = \left.\frac{\partial^2 W}{\partial \bar{I}_1^2}\right|_{\underset{\sim}{u}=0}\ , \text{ etc.} \tag{2.9}$$

It follows from (2.1), (2.2), (2.3), (2.6) and $(2.7)_{2,3,4}$, with the neglect of terms of higher degree than the first in the spatial derivatives of $\underset{\sim}{u}$, that

$$\begin{aligned} c_{ij} &= X_{i,\alpha}u_{j,\alpha} + X_{j,\alpha}u_{i,\alpha}\ , \\ i_1 &= 2X_{i,\alpha}u_{i,\alpha}\ , \quad i_2 = I_1 i_1 - C_{ij}c_{ji}\ . \end{aligned} \tag{2.10}$$

3. The incremental shear modulus

We now consider that the deformation $\underset{\sim}{\xi} \to \underset{\sim}{X}$ is a pure homogeneous deformation with principal extension ratios λ_1, λ_2, λ_3 and principal directions parallel to the axes of the rectangular cartesian coordinate system x. Thus*,

$$X_A = \lambda_A \xi_A \quad (A = 1,2,3) \; . \tag{3.1}$$

We suppose that a uniform simple shear of amount κ is superposed on this pure homogeneous deformation, the direction of shear being parallel to the unit vector $\underset{\sim}{\ell}$, say, and the plane of shear being parallel to the unit vectors $\underset{\sim}{\ell}$ and $\underset{\sim}{n}$, say, where $\underset{\sim}{n}$ is perpendicular to $\underset{\sim}{\ell}$. Then, in (2.6) $\underset{\sim}{u}$ is given by

$$\underset{\sim}{u} = \kappa(\underset{\sim}{n} \cdot \underset{\sim}{X})\underset{\sim}{\ell} \; . \tag{3.2}$$

With (3.1), equations (2.1), (2.2) and (2.3) yield

$$C_{AB} = \lambda_A^2 \delta_{AB} \; , \quad I_1 = \sum_B \lambda_B^2 \; , \quad I_2 = \sum_B \lambda_B^{-2} \; , \tag{3.3}$$

where δ_{AB} is the Kronecker delta and $\sum_B$ denotes summation over $B = 1,2,3$. The incompressibility of the material implies that

$$\lambda_1 \lambda_2 \lambda_3 = 1 \; . \tag{3.4}$$

Introducing (3.1), (3.2) and $(3.3)_{1,2}$ into (2.10), we obtain, with $\underset{\sim}{\ell} \cdot \underset{\sim}{n} = 0$ and (3.4),

* We shall assume that the Einstein summation convention is not applicable to upper case Latin subscripts.

$$c_{AB} = \kappa(\ell_A n_B \lambda_B^2 + \ell_B n_A \lambda_A^2) ,$$

$$i_1 = 2\kappa \sum_A \ell_A n_A \lambda_A^2 , \tag{3.5}$$

$$i_2 = 2\kappa\{\sum_B \lambda_B^2 \sum_A \ell_A n_A \lambda_A^2 - \sum_A \ell_A n_A \lambda_A^4\} = -2\kappa \sum_A \ell_A n_A \lambda_A^{-2} .$$

From $(3.3)_1$ and $(3.5)_1$, we obtain

$$(\underset{\sim}{C}^2)_{AB} = \lambda_A^4 \delta_{AB} ,$$
$$(\underset{\sim}{c}\underset{\sim}{C} + \underset{\sim}{C}\underset{\sim}{c})_{AB} = \kappa(\ell_B n_A \lambda_A^2 + \ell_A n_B \lambda_B^2)(\lambda_A^2 + \lambda_B^2) . \tag{3.6}$$

Introducing $(3.3)_{1,2}$, (3.5) and (3.6) into (2.8) we obtain

$$s_{AB} = 2\kappa \bar{s}_{AB} - p\delta_{AB} , \tag{3.7}$$

where

$$\begin{aligned} \bar{s}_{AB} = {} & W_1(\ell_A n_B \lambda_B^2 + \ell_B n_A \lambda_A^2) \\ & + W_2\{\ell_A n_B \lambda_A^{-2} + \ell_B n_A \lambda_B^{-2} - 2\delta_{AB} \sum_C \ell_C n_C \lambda_C^{-2}\} \\ & + 2\lambda_A^2 \delta_{AB}\Big[W_{11} \sum_C \ell_C n_C \lambda_C^2 + W_{12}\{(I_1 - \lambda_A^2) \sum_C \ell_C n_C \lambda_C^2 \\ & - \sum_C \ell_C n_C \lambda_C^{-2}\} - W_{22}(I_1 - \lambda_A^2) \sum_C \ell_C n_C \lambda_C^{-2}\Big] . \end{aligned} \tag{3.8}$$

In deriving this result, we use (3.4) and the relations

$$\sum_A \ell_A^2 = \sum_A n_A^2 = 1 , \quad \sum_A \ell_A n_A = 0 . \tag{3.9}$$

We note that the plane material surface which is normal to the unit vector $\underset{\sim}{n}$ after the pure homogeneous deformation remains normal to $\underset{\sim}{n}$ after the imposition of the simple shearing deformation and the area of an element of this surface is unchanged by this deformation. Accordingly, the change f in the component of the stress vector acting on this surface in the direction of the unit vector $\underset{\sim}{\ell}$ is given by

$$f = \sum_{A,B} s_{AB}\ell_A n_B = 2\kappa \sum_{A,B} \bar{s}_{AB}\ell_A n_B . \tag{3.10}$$

The incremental shear modulus, μ say, for the simple shear considered is given by

$$\mu = f/\kappa . \tag{3.11}$$

Then, introducing (3.8) into (3.10) and using the relations (3.9) and $(3.5)_2$, we obtain

$$\begin{aligned}\tfrac{1}{2}\mu = W_1 \sum_A \lambda_A^2 n_A^2 + W_2 \sum_A \lambda_A^{-2}\ell_A^2 \\ + 2\Big[W_{11}\big(\sum_A \lambda_A^2 \ell_A n_A\big)^2 - 2W_{12}\big(\sum_A \lambda_A^2 \ell_A n_A\big)\big(\sum_A \lambda_A^{-2}\ell_A n_A\big) \\ + W_{22}\big(\sum_A \lambda_A^{-2}\ell_A n_A\big)^2\Big] .\end{aligned} \tag{3.12}$$

4. Conditions for the incremental shear modulus to be positive

In this section we consider that the direction of the unit vector $\underset{\sim}{n}$ is fixed and we derive conditions for the incremental shear modulus to be positive for all choices of the direction of shear $\underset{\sim}{\ell}$ perpendicular to $\underset{\sim}{n}$.

Let $\underset{\sim}{a}$, $\underset{\sim}{b}$ be two perpendicular unit vectors in a plane normal to $\underset{\sim}{n}$. We choose their positive directions so that $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{n}$ forms a right-handed triad. Then,

$$\underset{\sim}{a}\cdot\underset{\sim}{a} = \underset{\sim}{b}\cdot\underset{\sim}{b} = \underset{\sim}{n}\cdot\underset{\sim}{n} = 1 , \quad \underset{\sim}{a}\cdot\underset{\sim}{b} = \underset{\sim}{a}\cdot\underset{\sim}{n} = \underset{\sim}{b}\cdot\underset{\sim}{n} = 0 , \quad \underset{\sim}{n} = \underset{\sim}{a}\times\underset{\sim}{b} . \tag{4.1}$$

Since $\underset{\sim}{\ell}$ is perpendicular to $\underset{\sim}{n}$, we may express it in the form

$$\underset{\sim}{\ell} = \underset{\sim}{a}\cos\theta + \underset{\sim}{b}\sin\theta . \tag{4.2}$$

Introducing (4.2) into (3.12), we obtain

$$\tfrac{1}{2}\mu = P\cos 2\theta + Q\sin 2\theta + R , \tag{4.3}$$

where

$$\begin{aligned} P &= W_2P_1 + 2W_{11}P_2 - 4W_{12}P_3 + 2W_{22}P_4 , \\ Q &= W_2Q_1 + 2W_{11}Q_2 - 4W_{12}Q_3 + 2W_{22}Q_4 , \\ R &= W_1R_0 + W_2R_1 + 2W_{11}R_2 - 4W_{12}R_3 + 2W_{22}R_4 , \end{aligned} \tag{4.4}$$

and $P_1, P_2, \ldots, R_4$ are defined by

$$P_1 = \tfrac{1}{2}\sum_A \lambda_A^{-2}(a_A^2 - b_A^2) , \quad R_1 = \tfrac{1}{2}\sum_A \lambda_A^{-2}(a_A^2 + b_A^2) ,$$

$$Q_1 = \sum_A \lambda_A^{-2}a_Ab_A ;$$

$$P_2 = \tfrac{1}{2}\{\sum_A \lambda_A^2 n_A a_A\}^2 - \tfrac{1}{2}\{\sum_A \lambda_A^2 n_A b_A\}^2 ,$$

$$Q_2 = \{\sum_A \lambda_A^2 n_A a_A\}\{\sum_A \lambda_A^2 n_A b_A\} ,$$

$$R_2 = \tfrac{1}{2}\{\sum_A \lambda_A^2 n_A a_A\}^2 + \tfrac{1}{2}\{\sum_A \lambda_A^2 n_A b_A\}^2 ;$$

$$P_3 = \tfrac{1}{2}\{\sum_A \lambda_A^2 n_A a_A\}\{\sum_A \lambda_A^{-2} n_A a_A\} - \tfrac{1}{2}\{\sum_A \lambda_A^2 n_A b_A\}\{\sum_A \lambda_A^{-2} n_A b_A\} ,$$

$$Q_3 = \tfrac{1}{2}\{\sum_A \lambda_A^2 n_A a_A\}\{\sum_A \lambda_A^{-2} n_A b_A\} + \tfrac{1}{2}\{\sum_A \lambda_A^2 n_A b_A\}\{\sum_A \lambda_A^{-2} n_A a_A\} ,$$

$$R_3 = \tfrac{1}{2}\{\sum_A \lambda_A^2 n_A a_A\}\{\sum_A \lambda_A^{-2} n_A a_A\} + \tfrac{1}{2}\{\sum_A \lambda_A^2 n_A b_A\}\{\sum_A \lambda_A^{-2} n_A b_A\} ; \qquad (4.5)$$

$$P_4 = \tfrac{1}{2}\{\sum_A \lambda_A^{-2} n_A a_A\}^2 - \tfrac{1}{2}\{\sum_A \lambda_A^{-2} n_A b_A\}^2 ,$$

$$Q_4 = \{\sum_A \lambda_A^{-2} n_A a_A\}\{\sum_A \lambda_A^{-2} n_A b_A\} ,$$

$$R_4 = \tfrac{1}{2}\{\sum_A \lambda_A^{-2} n_A a_A\}^2 + \tfrac{1}{2}\{\sum_A \lambda_A^{-2} n_A b_A\}^2 ;$$

$$R_0 = \sum_A \lambda_A^2 n_A^2 .$$

We can rewrite (4.3) as

$$\tfrac{1}{2}\mu = (P^2+Q^2)^{\frac{1}{2}}\sin(2\theta+\chi) + R , \qquad (4.6)$$

where

$$\chi = \tan^{-1}(P/Q) . \qquad (4.7)$$

It is apparent that the necessary and sufficient conditions for μ to be positive for all values of θ are

$$R > 0 \quad \text{and} \quad R^2 > P^2 + Q^2 . \qquad (4.8)$$

The stationary values of μ occur when

$$\theta = \frac{m\pi}{4} - \frac{\chi}{2} \quad (m = 1,3,5,7) . \qquad (4.9)$$

$m = 5,7$ correspond to reversal of the directions of $\underset{\sim}{\ell}$ corresponding to $m = 1,3$ respectively. We need therefore consider only the cases when $m = 1,3$. The two corresponding directions of $\underset{\sim}{\ell}$ are perpendicular. Whem $m = 1$, μ has a maximum value and when $m = 3$, μ has a minimum value. From (4.6) it is evident that these values are given by

$$\mu = 2\{R \pm (P^2+Q^2)^{\frac{1}{2}}\} . \tag{4.10}$$

5. Propagation of a plane sinusoidal wave

We now suppose that instead of subjecting the pure homogenously deformed material to a static infinitesimal simple shearing deformation, we propagate, in the direction $\underset{\sim}{n}$, a cosinusoidal shear wave of infinitesimal amplitude κ and angular frequency ω, polarized in the direction $\underset{\sim}{\ell}$. We note that for an incompressible material, the directions of propagation and polarization are necessarily perpendicular.

With the usual complex notation, the displacement field $\underset{\sim}{u}$ associated with this wave is given by

$$\underset{\sim}{u} = \kappa\underset{\sim}{\ell} \exp \imath\omega(S\underset{\sim}{n}\cdot\underset{\sim}{X} - t) . \qquad (5.1)$$

It is evident from (2.8) and (2.10) that the incremental stress associated with the deformation (5.1) is obtained from (3.7) by replacing κ by $\imath\omega S\kappa \exp \imath\omega(S\underset{\sim}{n}\cdot\underset{\sim}{X} - t)$, thus

$$s_{AB} = 2\imath\omega S\kappa\,\bar{s}_{AB} \exp \imath\omega(S\underset{\sim}{n}\cdot\underset{\sim}{X} - t) - p\delta_{AB} ; \qquad (5.2)$$

where $\bar{s}_{AB}$ is given by (3.8). We take the incremental hydrostatic pressure p in the form

$$p = \bar{p} \exp \imath\omega(S\underset{\sim}{n}\cdot\underset{\sim}{X} - t) , \qquad (5.3)$$

where $\bar{p}$ is independent of $\underset{\sim}{X}$ and t.

Bearing in mind the fact that the deformation $\underset{\sim}{\xi} \to \underset{\sim}{X}$ is homogeneous, we obtain the incremental equation of motion as

$$\sum_B \frac{\partial s_{AB}}{\partial X_B} = \rho\ddot{u}_A , \qquad (5.4)$$

where ρ is the material density.

Introducing (5.1), (5.2), (5.3) into (5.4), we obtain

$$2 \sum_{B} n_B \bar{s}_{AB} + \frac{i\bar{p}}{\kappa\omega S} n_A = \frac{\rho}{S^2} \ell_A \ . \tag{5.5}$$

Multiplying (5.5) throughout by ℓ_A and summing over $A = 1,2,3$, we have, with (3.9), (3.10) and (3.11),

$$S^2 = \rho/\mu \ , \tag{5.6}$$

where μ is given by (3.12).

6. Reality of the velocities of propagation of a wave

Equations (5.5), together with the equation

$$\sum_A \ell_A n_A = 0 \tag{6.1}$$

are equivalent to the equations (3.3) and (2.15) of [2] which were used to calculate the possible values of ρ/S^2. It was found in [2] that there are two such values which are given by the roots of the quadratic equation

$$\left(\frac{\rho}{S^2}\right)^2 - \beta\left(\frac{\rho}{S^2}\right) + \alpha = 0 , \tag{6.2}$$

where

$$\begin{aligned}
\alpha = {} & 4(\lambda_1^2 n_1^2 + \ldots)\{(\lambda_1^2 n_1^2 K_2 K_3 + \ldots) \\
& + W_1[n_2^2 n_3^2(\lambda_2^2 - \lambda_3^2)^2 M_1 + \ldots]\} \\
& + 4W_2[\lambda_2^2\lambda_3^2 n_2^2 n_3^2(\lambda_2^2 - \lambda_3^2)^2 M_1 + \ldots] \\
& + 16 n_1^2 n_2^2 n_3^2(\lambda_2^2 - \lambda_3^2)^2(\lambda_3^2 - \lambda_1^2)^2(\lambda_1^2 - \lambda_2^2)^2(W_{11}W_{22} - W_{12}^2) , \\
\beta = {} & 2\{[K_1(\lambda_2^2 n_2^2 + \lambda_3^2 n_3^2) + \ldots] \\
& + [(\lambda_2^2 - \lambda_3^2)^2 n_2^2 n_3^2 M_1 + \ldots]\} ,
\end{aligned} \tag{6.3}$$

and K_A, M_A $(A = 1,2,3)$ are defined by

$$K_A = W_1 + \lambda_A^2 W_2 , \quad M_A = 2(W_{11} + 2\lambda_A^2 W_{12} + \lambda_A^4 W_{22}) . \tag{6.4}$$

In (6.3), the dots indicate terms obtained from those given by cyclic permutation of the subscripts 1,2,3 on the λ's, n's, K's and M's.

As was seen in [2], the necessary and sufficient conditions that the velocities of propagation, parallel to the unit vector

$\underset{\sim}{n}$, of sinusoidal waves of infinitesimal amplitude, be real are

$$\alpha > 0 \ , \quad \beta > 0 \ , \quad \beta^2 - 4\alpha \geq 0 \ . \tag{6.5}$$

In [2] the following expression was also given for $\beta^2 - 4\alpha$:

$$\begin{aligned}
\beta^2 - 4\alpha = {} & 4W_2^2\{[\lambda_1^4 n_1^4(\lambda_2^2 - \lambda_3^2)^2 + \ldots] \\
& + 2[\lambda_2^2\lambda_3^2 n_2^2 n_3^2(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2) + \ldots]\} \\
& + 8W_2\{[(\lambda_2^2 - \lambda_3^2)^2 n_2^2 n_3^2 M_1\{\lambda_2^2(\lambda_1^2 - \lambda_3^2)(n_1^2 + n_2^2) \\
& + \lambda_3^2(\lambda_1^2 - \lambda_2^2)(n_1^2 + n_3^2)\}] + \ldots\} \\
& + 4\{[(\lambda_2^2 - \lambda_3^2)^2 n_2^2 n_3^2 M_1 + \ldots]^2 \\
& + 16n_1^2 n_2^2 n_3^2(\lambda_2^2 - \lambda_3^2)^2(\lambda_3^2 - \lambda_1^2)^2(\lambda_1^2 - \lambda_2^2)^2(W_{12}^2 - W_{11}W_{22})\} \ .
\end{aligned} \tag{6.6}$$

By substituting in (6.6) the expression for $\underset{\sim}{n}$ given in (4.1) and the expressions (6.4) for K_A and M_A , it can be shown, after lengthy algebraic manipulation, that

$$\beta^2 - 4\alpha = 16(P^2+Q^2) \ , \tag{6.7}$$

where P and Q are defined by $(4.4)_{1,2}$ with (4.5). Thus, the relation $\beta^2 - 4\alpha \geq 0$ in (6.5) is automatically satisfied.

In a similar manner it can be shown that

$$\beta = 4R \ . \tag{6.8}$$

The validity of (6.7) and (6.8) can, of course, be verified pragmatically by choosing arbitrary numerical values for W_1, W_2, W_{11}, W_{12}, W_{22} and arbitrary values for a_A, b_A, n_A, λ_A which satisfy the relations (4.1) and (3.4). Then $\beta^2 - 4\alpha$ and β can be calculated by using (6.6) and (6.8) and P^2+Q^2

and R can be calculated from (4.4) and (4.5).

From (6.7) and (6.8) it follows that

$$\alpha = 4(R^2 - P^2 - Q^2) \ . \tag{6.9}$$

Equations (6.7), (6.8) and (6.9) imply that the conditions (6.5) for reality of the wave velocities are precisely the same as the conditions (4.8) for the incremental shear modulus to be positive.

It was shown in [3] that if we restrict the direction of propagation of the wave to lie in a principal plane of the underlying pure homogeneous deformation, then the necessary and sufficient conditions that the velocities of propagation be real are

$$K_A > 0 \quad \text{and} \quad \left(I_1 - \lambda_A^2 - \frac{2}{\lambda_2}\right)M_A + K_A > 0 \quad (A = 1,2,3) \ . \tag{6.10}$$

It was shown in [1] that these are also the conditions that the incremental shear modulus be positive if the plane of shear is a principal plane. It was further shown in [2] that the conditions (6.10) imply that the relation $(6.5)_2$, and hence the relation $(4.8)_1$, is satisfied for arbitrary choice of the unit vector $\underset{\sim}{n}$. However, it was shown in [4] that they do not imply that the relation $(6.5)_1$, and equivalently the relation $(4.8)_2$, is satisfied for arbitrary $\underset{\sim}{n}$.

7. Directions of polarization of the wave

From (6.2) it follows, with (6.7) and (6.8), that

$$\frac{\rho}{S^2} = \frac{1}{2}\{\beta \pm (\beta^2 - 4\alpha)^{\frac{1}{2}}\} = 2\{R \pm (P^2 + Q^2)^{\frac{1}{2}}\} \tag{7.1}$$

for a wave propagating in the direction of $\underset{\sim}{n}$. We see from (5.6) that ρ/S^2 is the value of the incremental shear modulus for shear in the direction of polarization of the wave, the plane of shear being parallel to $\underset{\sim}{n}$.

From (4.10) it is seen that the expressions on the right hand side of (7.1) are the stationary values of the incremental shear modulus for specified $\underset{\sim}{n}$. It follows that the directions of polarization of the sinusoidal waves, which can be propagated parallel to $\underset{\sim}{n}$, are the directions for which the incremental shear modulus, for directions of shear perpendicular to $\underset{\sim}{n}$ and plane of shear parallel to $\underset{\sim}{n}$, has stationary values.

Acknowledgement

The results presented in this paper were obtained in the course of research sponsored by the Office of Naval Research under Contract No. N00014-76-C-0235 with Lehigh University.

References

[1] K.N. Sawyers and R.S. Rivlin, Developments in Mechanics 7, Proc. 13th Midwestern Mechanics Conference 321 (1973).

[2] K.N. Sawyers and R.S. Rivlin, Journal of Applied Mathematics and Physics 28, 1045 (1977).

[3] K.N. Sawyers and R.S. Rivlin, Int. J. Solids Structures 9, 607 (1973).

[4] K.N. Sawyers and R.S. Rivlin, Mechanics Research Communications (in the press).

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER CAM-100-32	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) The Incremental Shear Modulus in an Incompressible Elastic Material		5. TYPE OF REPORT & PERIOD COVERED Technical Report 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) K.N. Sawyers and R.S. Rivlin		8. CONTRACT OR GRANT NUMBER(s) N00014-76-C-0235
9. PERFORMING ORGANIZATION NAME AND ADDRESS Lehigh University Center for the Application of Mathematics Bethlehem, Pa. 18015		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Arlington, Virginia		12. REPORT DATE September, 1978 13. NUMBER OF PAGES 18
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) Same as above		15. SECURITY CLASS. (of this report) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Distribution of this document is unlimited		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Finite elasticity, material stability, Hadamard criterion		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The incremental shear modulus is calculated for infinitesimal shear of an incompressible isotropic elastic material in a state of pure homogeneous deformation. Necessary and sufficient conditions that it be positive are obtained. These conditions are related to the Hadamard conditions for propagation of a wave of infinitesimal amplitude in the deformed material.		

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)